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Multi-region heat conduction problems by boundary element method

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Abstract

This paper is concerned with boundary element solution of two dimensional steady-state heat conduction problems in multi-regions. In the proposed method, each region with different heat transfer properties is considered as a piecewise homogeneous in a heterogeneous system. The solution scheme akin to a finite difference method or finite element method sweeping treatment is adopted. The generated integral equation for a typical region is swept to obtain the system matrix for all regions in a single step without considering compatibility conditions explicitly at interfaces. In the case of linear or higher order elements, the non-square global system matrix is solved by the singular value decomposition method. Multi-region test problems for square and circular domains are considered and numerical results are presented.

1. Introduction

The thermal performance of composite materials is of critical concern for engineers. To know temperature distribution in each region in such media is particularly important for numerous applications in heat transfer problems from rocket thrust chamber liners to the fuel elements for nuclear reactors and many other fields.

Multi-region subject has been extensively studied analytically and numerically in classical monographs (see for example Refs. [1,2]). Among the numerical methods, finite difference and finite element methods are well-established. Recent advances on the subject are also reported in the boundary element method (BEM) literature for different purposes. Huang and Shaw used collocation technique in Treftz method, based on asymptotic expansions of particular solutions, to couple subregions [3]. This technique generates additional equations in the final algebraic system. Later, Portella and Charafi applied Trefftz boundary element method to potential problems in arbitrarily shaped domains [4]. For parallel computation reasons, Kamiya et al. used the Uzawa and Schwarz methods for virtual internal boundaries of the domain [5]. Then, Meric solved Laplace's equation iteratively by an optimization based domain decomposition method [6].

This paper presents a boundary element method (BEM) for solving temperature distribution in multi-region heat conduction problems [7]. The applied technique here is similar to a sweeping treatment of finite difference method or finite element method for multi-region problems. The system matrix representing the whole domain is generated in a single step without considering compatibility conditions explicitly at interfaces. This avoids extra numerical calculation step as faced in classical BEM coupling. The obtained linear equation system is solved by the singular value decomposition method (SVD).

2. Theory

Heat conduction equation in steady-state:

$$\nabla^2 T_s(\vec{r}) = -\frac{1}{k_s} Q_s(\vec{r}), \tag{1}$$

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Nomenclature

A	the system matrix
b	the known vector
G_s	the local Green's function for region s
k	thermal conductivity, W m^{-1} K^{-1}
$\vec{n}$	surface normal
N^c	shape function for the node c
$\bar{q}$	prescribed flux, W m^{-2}
Q	heat source, W m^{-3}
$\vec{r}$	position vector for field point
$\overline{T}$	prescribed temperature, K
T	temperature, K
u	an orthogonal matrix
v	an orthogonal matrix
w	a diagonal matrix
x	the unknown vector

Greek symbols

Γ	closed surface
$\vec{\rho}$	position vector for source point

Subscripts

i	the number of interfaces between regions
s	the number of homogeneous regions

Superscript

c	the node number

where k is the thermal conductivity, T is the temperature, Q is the heat source, s represents homogenous regions for $s = 1, \ldots, S$. This equation may be applied to a domain bounded by a closed smooth surface Γ, as shown in Fig. 1. Boundary conditions are given on the sections of the boundary Γ as follows:

$$T = \overline{T} \text{ on } \Gamma_1, \quad q = \bar{q} \text{ on } \Gamma_2, \tag{2}$$

where $\overline{T}$ and $\bar{q}$ are prescribed temperature and flux, respectively. For two neighbouring regions s and $s+1$, interface conditions:

$$\text{at } \vec{r} = \vec{r}_i \quad T_s(\vec{r}_i) = T_{s+1}(\vec{r}_i), \tag{3a}$$

$$\text{at } \vec{r} = \vec{r}_i \quad k_s \frac{dT_s(\vec{r})}{dn} = k_{s+1} \frac{dT_{s+1}(\vec{r})}{dn}, \tag{3b}$$

denoting C^0 and C^1 compatibility conditions and where i indices represent interfaces between homogenous regions for $i = 1, \ldots, I$—the number of total interfaces and $s = 1, \ldots, S$—the number of subregions. Now for the region s, the local Green's function

$$\nabla^2 G_s(\vec{r}, \vec{\rho}) = -\frac{1}{k_s} \delta(\vec{r} - \vec{\rho}), \tag{4}$$

where $\vec{r}$ and $\vec{\rho}$ are field and source points, respectively.

Multiplying Eqs. (1) and (4) with G_s and T_s, respectively, and subtracting from each other and integrating over the subdomain V_s, and integrating by parts, boundary integral equation form of Eq. (1) can be written in the general form (see, for example, [8,10])

$$T_s(\vec{\rho}) = \int_{V_s} d\vec{r} Q_s(\vec{r}) G_s(\vec{r}, \vec{\rho}) + k_s \int_{\Gamma_s} d\Gamma \left[G_s(\vec{r}, \vec{\rho}) T_s \frac{\partial T_s(\vec{r})}{\partial n} - T_s(\vec{r}) \frac{\partial G_s(\vec{r}, \vec{\rho})}{\partial n} \right], \tag{5}$$

where $\vec{n}$ is the surface normal. The volume term in Eq. (5) can be transformed into surface integrals provided Q is a harmonic function and we can find such as U that satisfies [9]

$$\nabla^2 U(\vec{r}, \vec{\rho}) = -G(\vec{r}, \vec{\rho}), \tag{6}$$

then

$$\int_{V_s} d\vec{r} Q_s(\vec{r}) G_s(\vec{r}, \vec{\rho}) = \int_{\Gamma_s} d\Gamma \left[Q_s(\vec{r}) \frac{\partial U(\vec{r}, \vec{\rho})}{\partial n} - U(\vec{r}, \vec{\rho}) \frac{\partial Q_s(\vec{r})}{\partial n} \right]. \tag{7}$$

The fundamental solutions satisfy Eq. (4) in two dimensions

$$G_s(\vec{r}, \vec{\rho}) = -\frac{1}{2\pi k_s} \ln \vec{R}, \tag{8}$$

where $\overline{R} = \vec{r} - \vec{\rho}$. The boundary is represented with constant or linear elements. The Cartesian co-ordinates x_i of an arbitrary point of an element defined in terms of nodal co-ordinates x_i^c and shape functions can be calculated from

$$x_i = N^c x_i^c, \tag{9}$$

where c is the node number which ranges from 1 to 2, $i = 1$, 2 and N^c are the shape functions of elements defined. Each solution variable, temperature and flux, can

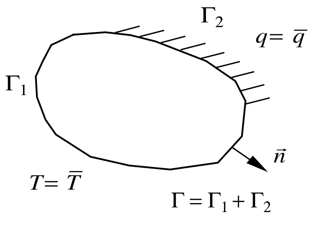

Fig. 1. Domain representation.

then be represented in terms of the same shape functions as follows:

$$T = N^c T^c, \quad q = N^c q^c, \tag{10}$$

where T^c and q^c are the nodal values of temperature and flux, respectively. Substituting the parametric representations of geometry, temperature and flux into Eq. (5), for a regionwise constant heat source, the boundary integral equation may then be written in the discretised form as

$$\begin{aligned} c(\vec{\rho})T_s(\vec{\rho}) &= \frac{1}{2\pi}\sum_{m=1}^{M}\frac{q_s^c}{k_s}\int_{\Gamma_m} N^c \ln|\vec{R}|\,\mathrm{d}\Gamma - \frac{1}{2\pi}\sum_{m=1}^{M} T_s^c \int_{\Gamma_m} N^c \frac{\vec{R}\cdot\vec{n}}{R^2}\,\mathrm{d}\Gamma \\ &\quad + \frac{1}{4}\sum_{m=1}^{M}\frac{Q_s}{k_s}\int_{\Gamma_m} N^c[\vec{R}\cdot\vec{n}(2\ln|\vec{R}|-1)]\,\mathrm{d}\Gamma, \end{aligned} \tag{11}$$

where $c(\vec{\rho})$ takes value between 0 and 1 depending on where $\vec{\rho}$ lies (see, for example, [10]). The accuracy of the BEM essentially depends on the accuracy of evaluation of integrals. Therefore, all integrals in Eq. (11) are calculated analytically. The details of the integration schemes are covered in Refs. [8–10].

In conventional techniques such as FDM or FEM, multi-domain problems are relatively easy to deal with. Each element is treated as a distinct entity having its own material properties. However, the continuity of flux across the regional interfaces is not enforced. This is because the FEM demand only temperature continuity between elements [9]. Therefore, FDM or FEM is naturally attracting more interests for multi-regional problems.

In the usual BEM approach in the literature, each subregion is treated separately to form the regional matrices. The regional matrices are processed according to known and unknown nodal values and eventually combined according to relevant compatibility conditions to arrive at the system matrix which represents whole domain. Here, some questions remain as to the efficiency and cost of the method. Much of difficulty arises from the fact that each subregion is handled separately. This scheme becomes quite expensive especially when the number of regions becomes quite large and the order of elements becomes higher. The main portion of the classical procedure for a constant element formulation is [8–10]

Do while $s = 1, \ldots, S$	* Do for region s *
Do while $j = 1, \ldots, N$	* Do for each node j *
Write Eq. (11) for the source at element j	* Eq. (11) for each node *
Enddo j	* $A_s x_s = b_s$ *
Enddo s	
Do while $i = 1, \ldots, I$	* Do for interface i *
Use C^0 compatibility explicitly combine $A_s x_s = b_s$ into $Ax = b$	
Enddo s	* $Ax = b$ *

The proposed scheme here is robust, allowing the multi-domain problems to be treated efficiently. The generated equation for a typical region, Eq. (11), is swept to obtain as a system matrix for all regions in a single step. For this purpose, each region is considered as a part of the full system with an appropriate node and element number assignment. Since subregion matrices are not formed individually here, the procedure is FEM (or FDM) like, with one exception that flux compatibility (C^1) is imposed implicitly as in classical BEM treatment. In the classical procedure, compatibility conditions are used in an extra second step to combine subdomain matrices into a final system matrix. Overall the procedure is desirably simplified but it costs resulting in not necessarily always a square system matrix as in the case of linear or higher order elements employed. The proposed procedure is:

Do while $s = 1, \ldots, S$	* Do for region s *
Do while $j = 1, \ldots, N$	* Do for each node j *
Write Eq. (11) for the source at element j	
Enddo j	
Enddo s	* $Ax = b$*

It is important to note that, for linear elements case, at corner nodes of the subregions as in Fig. 2, double nodes are used. This is due to the need of directional flux separations at these particular nodes. In the sweeping treatment, it is also worth mentioning that the sign of flux must be reversed if the interface normal is in the negative direction. A final system of linear algebraic equations is obtained as follows

$$\mathbf{AX} = \mathbf{b}, \tag{12}$$

where $\mathbf{A}$ is the resulting system matrix, $\mathbf{x}$ is the unknown vector in the form

$$\mathbf{x} = \begin{bmatrix} T_i \\ - \\ q_i \end{bmatrix},$$

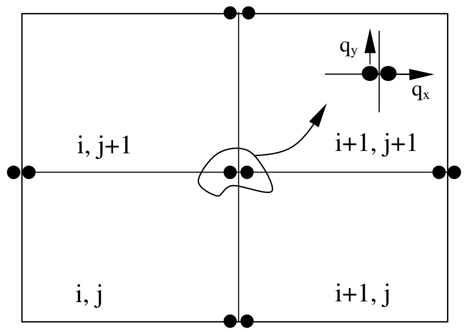

Fig. 2. Double node selection at corners for linear elements.

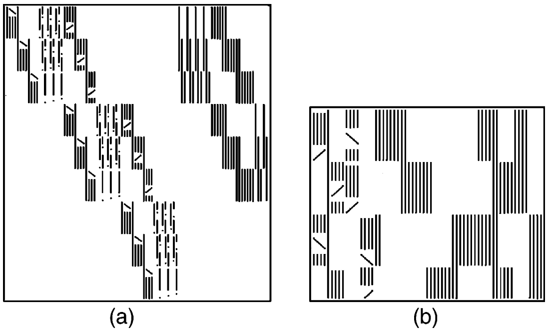

Fig. 3. The system matrices **A** showing the distribution of non-zero elements for test cases: (a) test case 1 (144×140), (b) test case 2 (64×63).

where $i = 1, \ldots, n$, and **b** is the known vector caused by the source and prescribed boundary values. The matrix, **A** is always sparse and non-symmetric in general. Typical appearances of the matrix **A** for test cases solved below are given in Fig. 3.

Since the matrix **A** is non-square for the case of linear elements employed, as a solution method, the singular value decomposition (SVD) method is used [11]. Let **u** and **v** are orthogonal matrices, then, the matrix **A** has the singular value decomposition in the form

$$\mathbf{A} = \mathbf{u}\mathbf{w}\mathbf{v}^{\mathrm{T}}, \tag{13}$$

where **w** is a diagonal matrix. Then the solution of Eq. (12) is given by

$$\mathbf{x} = \mathbf{v}\mathbf{w}^{-1}\mathbf{u}^{\mathrm{T}}\mathbf{b}. \tag{14}$$

3. Numerical results

3.1. Case 1: square domain

As a first case, a square domain $L \times L$ (18×18) is considered. The domain has nine (3×3) regions with their own thermal properties. The flux on the left and bottom sides are fixed at zero, while temperature are taken 100 at the top and 50 at the right sides of the domain. Table 1 shows the thermal conductivity coefficients of nine subregions.

The problem is modelled here with 16 equal length constant elements at each subregion. FDM method with a 24×24 (eight in each region) grid, employing mesh centered scheme is also used to asses the results of the BEM. The results along the domain center lines are compared in Figs. 4 and 5, where the agreement is good.

Table 1
Thermal conductivity coefficients of the subregions

i	*j* 1	2	3
1	0.60	0.70	0.80
2	0.70	0.50	0.60
3	0.80	0.60	0.50

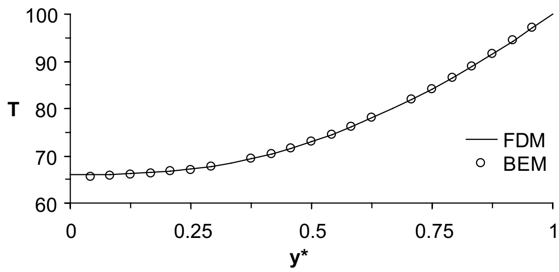

Fig. 4. Temperature profile along the vertical center line ($x^* = x/L = 0.5$).

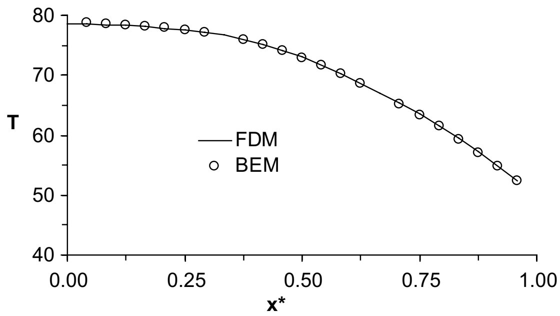

Fig. 5. Temperature profile along the horizontal center line ($y^* = y/L = 0.5$).

3.2. Case 2: cylindrical domain

The second test problem considered here is a circular domain with four subregions as shown in Fig. 6. The outer boundary of the domain is subjected to zero

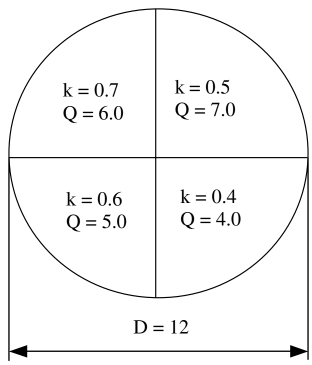

Fig. 6. Thermal conductivity and source values for the second case.

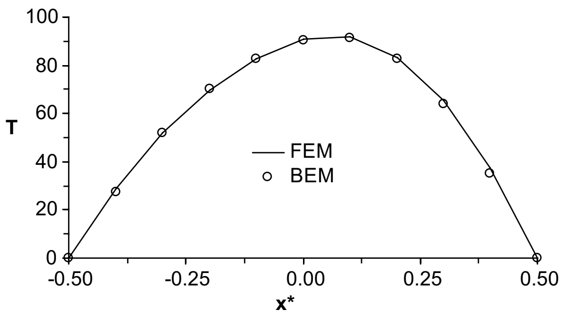

Fig. 7. Temperature profile at $y^* = y/D = 0$.

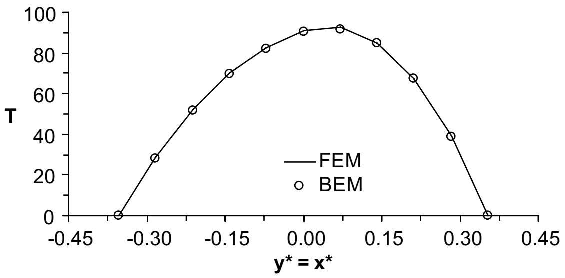

Fig. 8. Temperature profile at $y^* = x^*$.

temperature. 16 linear boundary elements in each subregion (total 44 elements) in the BEM mesh are employed to model the case. The FEM based on the mesh with 456 number of linear triangular elements and 241 nodes is used to solve the same case for comparison. The FEM is especially chosen, instead of the FDM, due to the singularity at the origin in the discretized equations of FDM. Figs. 7 and 8 show temperature profiles along $x = 0$ and $y = x$.

4. Conclusion

This work presents an accurate and efficient solution of steady-state heat conduction problems in multi-regions by the BEM. The methodology proposed here eliminates the disadvantage of BEM against other classical discretization techniques such as FDM and FEM. For this purpose, an algorithm to handle multi-region problems in heat conduction problems is presented. A straightforward approach to form global system matrix is employed and linear system is solved by the SVD technique. Two test problems in different geometries were studied by the proposed scheme and compared with FDM and FEM. It is shown that the results are in complete agreement and the proposed solution technique is quite efficient.

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References

[1] S. Nakamura, Computational Methods in Engineering and Science, Robert E. Krieger Publishing Company, Florida, 1986.
[2] N. Ozisik, Heat Conduction, John Wiley & Sons, New York, 1980.
[3] S.C. Huang, R.P. Shaw, The Trefftz method as an integral equation, Adv. Eng. Software 24 (1995) 57–63.
[4] A. Portela, A. Charafi, Trefftz boundary elements—multi-region formulations, Int. J. Numer. Meth. Eng. 45 (1999) 821–840.
[5] N. Kamiya, H. Iwase, E. Kita, Parallel implementation of boundary element method with domain decomposition, Eng. Anal. BEM 18 (1996) 209–216.
[6] R.A. Meric, Domain decomposition methods for Laplace's equation by the BEM, Commun. Numer. Meth. Eng. 16 (2000) 545–557.
[7] M.A. Atalay, E.D. Aydin, M. Aydin, Multi-region heat conduction problems by boundary element method, in: Proceedings of the First international Exergy, Energy and Environment Symposium, 13–17 July 2003, Izmir, pp. 123–127.
[8] P.W. Partridge, C.A. Brebbia, L. Wrobel, The Dual Reciporicity Boundary Element Method, CMP, Southampton, London, 1992.
[9] G.S. Gipson, Boundary Element Fundamentals, CMP, Southampton, 1987.
[10] A. El-Zafrany, Techniques of Boundary Element Method, Ellis Horwood, London, 1993.
[11] W.H. Press, S.A. Teukolosky, W.T. Vetterling, B.P. Flannery, Numerical Recipes in Fortran, Cambridge University Press, Cambridge, 1992.